

## Loewner driving functions for off-critical percolation clusters

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We numerically study the Loewner driving function  $U_t$  of a site percolation cluster boundary on the triangular lattice for  $p < p_c$ . It is found that  $U_t$  shows a drifted random walk with a finite crossover time. Within this crossover time, the averaged driving function  $\langle U_t \rangle$  shows a scaling behavior  $-(p_c - p)t^{(\nu+1)/2\nu}$  with a superdiffusive fluctuation whereas, beyond the crossover time, the driving function  $U_t$  undergoes a normal diffusion with Hurst exponent  $1/2$  but with the drift velocity proportional to  $(p_c - p)^\nu$ , where  $\nu = 4/3$  is the critical exponent for two-dimensional percolation correlation length. The crossover time diverges as  $(p_c - p)^{-2\nu}$  as  $p \rightarrow p_c$ .

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### I. INTRODUCTION

Loewner evolution has recently drawn much attention in the physics community because of the development of Schramm-Loewner evolution (SLE) [1], which has provided us a new tool for the study of two-dimensional (2-d) continuous phase transition. The basic device of SLE is a conformal mapping that transforms a motion along a nonintersecting 2-d curve into another motion along the real axis; *Loewner driving function* is a real function that represents this transformed motion. It turns out that, for a certain class of stochastic and conformally invariant curves in two dimensions, the driving function shows Brownian motion in one dimension. What makes SLE especially remarkable is that it gives us a method that describes all the geometrical properties of the curves through a single parameter of the Brownian motion, namely, the diffusion constant. The class of curves includes the self-avoiding walk, the uniform-spanning trees, the loop-erased random walk, and boundaries of critical clusters in various 2-d lattice models in physics such as percolation, Ising model,  $O(n)$  loop models, and Potts models (see [2] for review).

Being inspired by mathematically oriented development, people start using SLE formalism to test the conformal invariance by calculating Loewner driving functions obtained from 2-d curves in a number of physical systems, such as vorticity clusters and temperature isolines in turbulence [3], domain walls in 2-d spin glass [4], isoheight lines on growing solid surface [5], and nodal domains of chaotic maps [6].

In the study of physical systems, it is important to ask how driving functions may look like when the system departs from the critical point because in real life there are a number of sources that may drive a system away from it. There are some mathematical approaches to study the effects of off-criticality on SLE based on probability theory and conformal field theory [7,8], which mainly pursue mathematical consistency in the continuum limit, but general feature of the off-critical driving function is not known yet. In this Rapid Communication, we report our results of numeri-

cal simulations to study how the Loewner driving function deviates from the ideal Brownian motion when the system departs from the criticality in the case of percolation clusters.

### II. LOEWNER EVOLUTION

Let us start by reviewing basic elements of Loewner evolution briefly. Consider a nonintersecting continuous curve  $\gamma$  which starts from the origin and extends toward infinity in the upper half plane  $\mathbb{H}$ . We parametrize  $\gamma$  by  $t \geq 0$ , and denote a point on  $\gamma$  as  $\gamma_t$  with  $\gamma_0 = 0$ . A part of  $\gamma$  between  $\gamma_{t_1}$  and  $\gamma_{t_2}$  is represented by  $\gamma_{[t_1, t_2]}$ . It is known that, for given  $\gamma_{[0, t]}$ , there exists a unique conformal map  $g_t: \mathbb{H} \setminus \gamma_{[0, t]} \rightarrow \mathbb{H}$  that satisfies the condition

$$g_t(z) = z + a_t/z + O(|z|^{-2}) \quad \text{as } |z| \rightarrow \infty \quad (1)$$

with  $a_t \geq 0$  (Fig. 1). The parameter  $t$ , which we will call time, is now defined by  $t := a_t/2$ . The driving function  $U_t$  is defined by the image of  $\gamma_t$  by the map  $g_t$ :  $U_t := \lim_{z \rightarrow \gamma_t} g_t(z)$ . Then, it can be shown that  $g_t(z)$  satisfies the Loewner evolution,  $\partial_t g_t(z) = 2/(g_t(z) - U_t)$ .

Note that  $g_t(z)$  represents the complex electrostatic potential for the equipotential boundary of  $\gamma_{[0, t]}$  and the real axis [9], then one can see that the time  $a_t = 2t$  is equal to  $p/2\pi$ , where  $p$  is the 2-d dipole moment induced by  $\gamma_{[0, t]}$  on a flat electrode. In this electrostatic picture,  $U_t$  is just given by the charge induced by  $\gamma_{[0, t]}$  along the right side of  $\gamma_{[0, t]}$  and the positive part of the real axis in the unit system with  $\epsilon_0 = 1$  (Fig. 1).

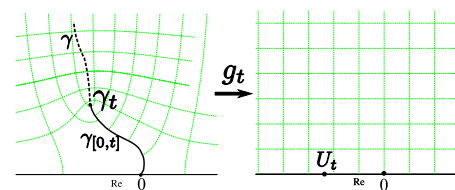


FIG. 1. (Color online) A schematic diagram of conformal mapping.  $g_t$  maps  $\mathbb{H} \setminus \gamma_{[0, t]}$  to  $\mathbb{H}$  with  $\gamma_t$  to  $U_t$  on the real axis. The deformed grid in the left panel can be regarded as equipotential and electric force lines, and is mapped to the straight grid in the right panel.

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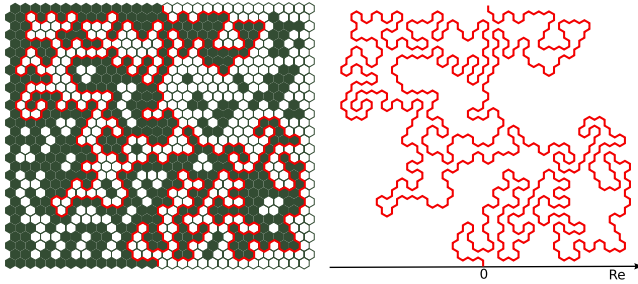


FIG. 2. (Color online) Percolation on the triangular lattice. The left panel shows a whole system with left (right) half of the system boundary being occupied (unoccupied). The right panel shows only the cluster boundary that starts from the origin.

Schramm has shown that  $U_t$  becomes a Brownian motion if the curve  $\gamma$  is a conformally invariant random curve with the domain Markov property [1]. This means that any properties of the curve such as the fractal dimension are determined solely by the diffusion constant of the Brownian motion.

### III. SIMULATIONS

We performed numerical simulations to obtain  $U_t$  for cluster boundaries of the site percolation on the triangular lattice with the occupation probability  $p \leq p_c = 0.5$ . Percola-

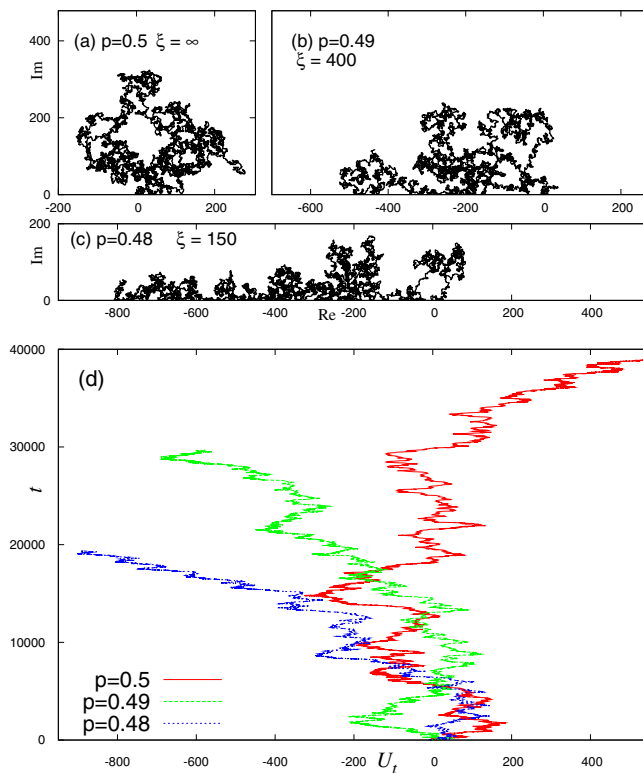


FIG. 3. (Color online) Examples of cluster boundaries (a)-(c) and their driving functions (d) for  $p=0.5$ ,  $0.49$ , and  $0.48$ . The boundaries are shown up to  $N=20000$  steps. The lattice spacing is set to be unity. The correlation lengths are  $\xi=400$  and  $150$  for  $p=0.49$  and  $0.48$ , respectively.

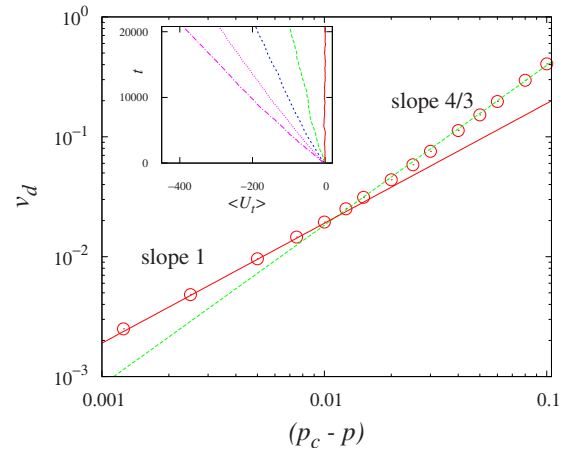


FIG. 4. (Color online) Drift velocity  $v_d$  vs  $(p_c - p)$ . The drift velocities of the driving function averaged up to  $t=21000$  are plotted in the logarithmic scale as a function of  $(p_c - p)$ . There is a crossover around  $(p_c - p) \sim 0.01$  between the two regimes. The solid (dashed) line with the slope  $1(4/3)$  shows the limiting behavior in the small (large)  $p_c - p$  regime. The inset shows  $t$  vs  $\langle U_t \rangle$  for  $p=0.5, 0.4975, 0.495, 0.4925,$  and  $0.49$  (from right to left). Each line represents average behavior over 4000 samples.

tion clusters are generated in the rectangular system. To ensure the cluster boundaries to extend from the origin in the upper half plane, the left (right) half of the boundary sites are set to be occupied (unoccupied), and the boundaries are defined along the edges of the dual lattice, i.e., the honeycomb lattice [2] (Fig. 2). We use only the part of cluster boundaries that never touches the peripheries except for the bottom side. For each boundary curve, we numerically generate the conformal map  $g_t(z)$  and compute the driving function  $U_t$  using the zipper algorithm [10,11].

Figures 3(a)–3(c) show typical examples of boundaries  $\gamma$  up to 20 000 steps at  $p=0.5$ ,  $0.49$ , and  $0.48$ , and Fig. 3(d) shows corresponding driving functions  $U_t$  [12]. For  $p=0.5$ , the boundary extends into  $\mathbb{H}$  indefinitely around the origin once it hits an infinite cluster. For  $p < p_c$ , the boundary tends to extend toward left in larger scale, but it can hardly be distinguished from that at  $p=p_c$  within the scale of correlation length  $\xi$  with a finite lattice constant. Accordingly,  $U_t$  wander around the origin at  $p=p_c$  whereas it drifts toward  $-\infty$  for large  $t$  for  $p < p_c$ . Note that the time does not increase uniformly with the step along a boundary because the time is proportional to the dipole moment induced by  $\gamma_{[0,t]}$ .

The averaged behavior of  $U_t$  is shown for several values of  $p$  in the inset of Fig. 4, where one can see  $\langle U_t \rangle$  appears to drift at a constant velocity  $v_d$ . In Fig. 4, the drift velocity  $v_d$  averaged up to  $t=21\,000$  over 4000 samples is plotted against  $p_c - p$  in the logarithmic scale;  $v_d$  is shown to be proportional to  $(p_c - p)$  for  $(p_c - p) \leq 0.01$ ,

$$v_d \sim (p_c - p), \quad (2)$$

whereas it behaves as

$$v_d \sim (p_c - p)^\nu \quad (3)$$

with  $\nu \approx 4/3$  for  $p_c - p \geq 0.01$ .

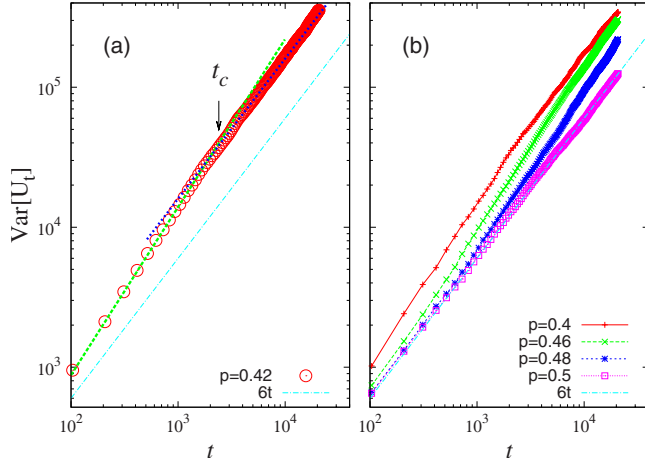


FIG. 5. (Color online)  $\text{Var}[U_t]$  vs  $t$  in the logarithmic scale. (a) The plot for  $p=0.42$ . The behavior changes from superdiffusive to diffusive around the crossover time  $t_c$  indicated by the arrow. The behavior for  $p=p_c$ ,  $\text{Var}[U_t]=6t$ , is plotted for comparison. (b) The plots for several values of  $p$ . The crossover time  $t_c$  becomes larger as we approach  $p_c$ .

The time dependence of the variance  $\text{Var}[U_t]$  is shown in the logarithmic scale in Fig. 5. As one can see in the plot for  $p=0.42$  in Fig. 5(a), there are two time regimes: the superdiffusion regime and the normal diffusion regime,

$$\text{Var}[U_t] \sim \begin{cases} t^\alpha & \text{for } t \ll t_c \\ t & \text{for } t \gg t_c \end{cases} \quad (4)$$

with the exponent  $\alpha > 1$  and the crossover time  $t_c$ . The plots for several values of  $p$  in Fig. 5(b) shows the tendency that the exponent  $\alpha$  decreases to 1 and the crossover time  $t_c$  increases as  $p \rightarrow p_c$ . The estimated values for  $t_c$  are plotted against  $(p_c - p)$  in the logarithmic scale by filled circles in Fig. 6(a).

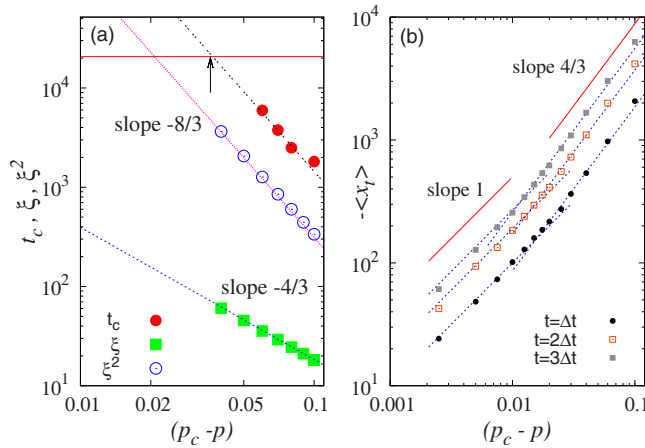


FIG. 6. (Color online) (a)  $t_c$ ,  $\xi$ , and  $\xi^2$  vs  $(p_c - p)$  in the logarithmic scale. The horizontal solid line at  $t=21000$  indicates the simulation range. (b)  $-\langle x_t \rangle$  vs  $(p_c - p)$  for  $t=\Delta t, 2\Delta t$ , and  $3\Delta t$  with  $\Delta t=5200$ , in the logarithmic scale. Each data point represents average over 3000 samples.

#### IV. DISCUSSIONS

Our results can be understood as in the following. First, we consider the curve of  $p=p_c$ . In this case,  $\gamma_t$  explores along a critical boundary, which extends indefinitely without bias. Consider a part of the curve  $\gamma_{[0,t]}$ . This may look like a blob, whose size we denote by  $l_t$ . The time  $t$  that corresponds to the blob can be estimated as

$$t \sim l_t^2, \quad (5)$$

using the electrostatic analogy; The time  $t$  is proportional to the dipole moment  $p$  induced by  $\gamma_{[0,t]}$  attached to a flat electrode, and both the induced charge and the charge displacement are of order of  $l_t$ . On the other hand,  $U_t$  may be approximated to be  $x_t := \text{Re } \gamma_t$  because  $U_t$  is determined from the number of electric force lines landing on the right side of  $\gamma_{[0,t]}$  and the positive part of real axis. Since  $x_t$  wanders along the blob of size  $l_t$ , one can see that  $U_t$  undergoes normal diffusion without drift:

$$U_t \sim l_t \sim \sqrt{t}. \quad (6)$$

For  $p < p_c$ , the curve  $\gamma$  is the boundary of finite off-critical clusters that are connected by the bottom. Since typical size of each cluster is the percolation correlation length  $\xi$ , the curve  $\gamma$  looks like a chain of the blobs of size  $\xi$  [see, for example, Fig. 3(c)]. Within each blob,  $\gamma$  is almost like a critical boundary but with a bias toward the left; the strength of the bias is proportional to  $p_c - p$ .

There should be two time regimes: the short time regime where  $\gamma_t$  is still in the first blob, and the long time regime where  $\gamma_t$  is traveling over blobs. The crossover time  $t_c$  between the two regimes is the time that  $\gamma_t$  goes over the first blob of size  $\xi$ ; therefore, it is estimated as  $t_c \sim \xi^2$  as in the case of critical blob. Within the short time regime,  $\gamma_t$  explores in the first blob that looks almost like a critical blob in the smaller length scale than  $\xi$  but with a small bias toward left. We assume the scaling behavior  $\langle x_t \rangle \sim -\xi(t/t_c)^\beta$  with an exponent  $\beta$ . If we determine the exponent so that the effect of bias should be proportional to  $(p_c - p)$  for a fixed  $t$ , we obtain  $\beta = (\nu + 1)/2\nu = 7/8$ , using the critical exponent  $\nu = 4/3$  for the correlation length of the 2-d percolation. On the other hand, the behavior of  $\langle x_t \rangle$  in the long time regime may be obtained by considering the situation where  $\gamma_t$  is at  $n$ th blob, i.e.,  $\langle x_t \rangle \sim n\xi$ . The time corresponding to this is  $t \sim n\xi^2$  because the dipole moment induced at each blob is of order of  $\xi^2$ , with which we can express  $\langle x_t \rangle$  as  $t/\xi$ . With all these argument, we finally obtain

$$\langle x_t \rangle \sim \begin{cases} -(p_c - p)t^{(\nu+1)/2\nu} & (t \leq t_c) \\ -(p_c - p)^\nu t & (t \geq t_c) \end{cases} \quad (7)$$

with  $t_c \sim (p_c - p)^{-2\nu}$ .

Within the approximation  $U_t \sim x_t$ , this is consistent with our results for the drift velocity  $v_d$  in Fig. 4; the slope we obtained for  $p_c - p \geq 0.01$  is very close to  $\nu = 4/3$ . The linear dependence for  $p_c - p \leq 0.01$  should correspond to the short time behavior. The crossover around  $p_c - p \approx 0.01$  is due to the fact that our simulation time length is not long enough in comparison with  $t_c$ . If we can simulate longer time, the crossover value of  $p_c - p$  will be smaller. The expected weak

nonlinear  $t$  dependence in  $x_t$  is difficult to distinguish from the linear behavior numerically; actually, one might notice slight convexity of the plots in the inset of Fig. 4.

In order to check further consistency of the data with our interpretation, we plot  $t_c$  for  $\text{Var}[U_t]$ ,  $\xi$ , and  $\xi^2$  against  $p_c - p$  in Fig. 6(a). The horizontal solid line indicates the time range of our simulations. The plotted range of the crossover time (filled circles) in  $\text{Var}[U_t]$  is too narrow to determine its behavior in a reliable way, but it seems consistent with that of  $\xi^2$  (open circles). The value of  $p_c - p$  where the extrapolated  $t_c$  of  $\text{Var}[U_t]$  reaches the simulation time is about 0.04 (arrow), which is larger than the crossover point in  $v_d$  by the factor 4; this discrepancy may come from the difference between  $v_d$  and  $\text{Var}[U_t]$ , or simply due to the uncertainty in the estimates. In Fig. 6(b),  $-\langle x_t \rangle$  for several values of  $t$  are plotted against  $(p_c - p)$ . From Eq. (7), we expect linear dependence on  $(p_c - p)$  for  $(p_c - p) \lesssim t^{-3/8}$  and  $(p_c - p)^{4/3}$  dependence beyond that; the data seem to be consistent.

With these observations, we interpret our results as follows. For an off-critical percolation cluster boundary, the driving function  $U_t$  undergoes a random walk with drift. The drift velocity  $v_d$  appears to be proportional to  $(p_c - p)$  when  $(p_c - p) \lesssim 0.01$ , but it is because time lengths of our simulations are finite; With the approximation  $U_t \sim x_t$ , the averaged driving function should be given by Eq. (7). Within the crossover time, the random walk is superdiffusive with exponent  $\alpha > 1$ , which decreases toward 1 as  $p \rightarrow p_c$ . Beyond the crossover time, the fluctuation around the drift motion is normal diffusion with larger diffusion constant than that at  $p = p_c$ .

Finally, let us discuss the scaling limit where the lattice constant  $a \rightarrow 0$  with keeping the correlation length  $\xi$  constant. Our argument to derive Eq. (7) holds for any  $\xi$  and  $a$  as long as  $\xi \gg a$ ; thus, if we scale the variables as  $\tilde{x}_t \equiv x_t / \xi$ ,  $\tilde{U}_t \equiv U_t / \xi$ , and  $\tilde{t} \equiv t / \xi^2$ , then we obtain the corresponding equation for the scaled variables, i.e., the same as Eq. (7) but without the factor of  $(p_c - p)$ .

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- [12] Note that, under the employed boundary condition, the finite system size effect comes into the problem as the limitation that the available length of  $\gamma$  is finite.